COORDINATE SFERICHE

vettore posizione: $\mathbf{r} = r\sin\theta\cos\phi\,\hat{\mathbf{x}} + r\sin\theta\sin\phi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}}$

trasformazione:
$$r = \sqrt{x^2+y^2+z^2}$$
$$\theta = \cos^{-1}\left(z/\sqrt{x^2+y^2+z^2}\right)$$
$$\phi = \tan2^{-1}(y,x)$$
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

elemento di volume: $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

fattori di scala: $h_r = \left|\frac{\partial\mathbf{r}}{\partial r}\right| = 1,\ h_\theta = \left|\frac{\partial\mathbf{r}}{\partial\theta}\right| = r,\ h_\phi = \left|\frac{\partial\mathbf{r}}{\partial\phi}\right| = r\sin\theta$

base locale:
$$\hat{\mathbf{r}}(\theta,\phi) = \sin\theta\,\cos\phi\,\hat{\mathbf{x}} + \sin\theta\,\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}}(\theta,\phi) = \cos\theta\,\cos\phi\,\hat{\mathbf{x}} + \cos\theta\,\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\phi}}(\phi) = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}}(\theta,\phi) + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}}(\theta,\phi) - \sin\phi\,\hat{\boldsymbol{\phi}}(\phi)$$
$$\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}}(\theta,\phi) + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}}(\theta,\phi) + \cos\phi\,\hat{\boldsymbol{\phi}}(\phi)$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}}(\theta,\phi) - \sin\theta\,\hat{\boldsymbol{\theta}}(\theta,\phi)$$

campo scalare: $f(r,\theta,\phi)$

gradiente: $\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\,\hat{\mathbf{r}}(\theta,\phi) + \frac{1}{r}\frac{\partial f}{\partial\theta}\,\hat{\boldsymbol{\theta}}(\theta,\phi) + \frac{1}{r\sin\theta}\,\frac{\partial f}{\partial\phi}\,\hat{\boldsymbol{\phi}}(\phi)$

c. vett.: $\mathbf{F}(r,\theta,\phi) = F_r(r,\theta,\phi)\,\hat{\mathbf{r}}(\theta,\phi) + F_\theta(r,\theta,\phi)\,\hat{\boldsymbol{\theta}}(\theta,\phi) + F_\phi(r,\theta,\phi)\,\hat{\boldsymbol{\phi}}(\phi)$

divergenza: $\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,F_\theta\right) + \frac{1}{r\sin\theta}\,\frac{\partial F_\phi}{\partial\phi}$

rotore:
$$\boldsymbol{\nabla}\times\mathbf{F} = \left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,F_\phi\right) - \frac{1}{r\sin\theta}\frac{\partial F_\theta}{\partial\phi}\right]\hat{\mathbf{r}}(\theta,\phi) + \left[\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rF_\phi\right)\right]\hat{\boldsymbol{\theta}}(\theta,\phi) + \left[\frac{1}{r}\frac{\partial}{\partial r}\left(rF_\theta\right) - \frac{1}{r}\frac{\partial F_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}(\phi)$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$\nabla^2\mathbf{F} = \left[\nabla^2F_r - \frac{2F_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(\sin\theta\,F_\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial F_\phi}{\partial\phi}\right]\hat{\mathbf{r}}(\theta,\phi) + \left[\nabla^2F_\theta - \frac{F_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\phi}{\partial\phi} + \frac{2}{r^2}\frac{\partial F_r}{\partial\theta}\right]\hat{\boldsymbol{\theta}}(\theta,\phi) + \left[\nabla^2F_\phi - \frac{F_\phi}{r^2\sin^2\theta} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\theta}{\partial\phi} + \frac{2}{r^2\sin\theta}\frac{\partial F_r}{\partial\phi}\right]\hat{\boldsymbol{\phi}}(\phi)$$

advezione: $\mathbf{a}\cdot\boldsymbol{\nabla} f = a_r\frac{\partial f}{\partial r} + \frac{a_\theta}{r}\frac{\partial f}{\partial\theta} + \frac{a_\phi}{r\sin\theta}\frac{\partial f}{\partial\phi}$

$$(\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{F} = \left[\mathbf{a}\cdot\boldsymbol{\nabla}F_r - \frac{a_\theta F_\theta + a_\phi F_\phi}{r}\right]\hat{\mathbf{r}}(\theta,\phi) + \left[\mathbf{a}\cdot\boldsymbol{\nabla}F_\theta + \frac{a_\theta F_r - \cot\theta\,a_\phi F_\phi}{r}\right]\hat{\boldsymbol{\theta}}(\theta,\phi) + \left[\mathbf{a}\cdot\boldsymbol{\nabla}F_\phi + \frac{a_\phi(F_r + \cot\theta\,F_\theta)}{r}\right]\hat{\boldsymbol{\phi}}(\phi)$$
$$= \left[a_r\frac{\partial F_r}{\partial r} + \frac{a_\theta}{r}\left(\frac{\partial F_r}{\partial\theta} - F_\theta\right) + \frac{a_\phi}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - F_\phi\right)\right]\hat{\mathbf{r}}(\theta,\phi) + \left[a_r\frac{\partial F_\theta}{\partial r} + \frac{a_\theta}{r}\left(\frac{\partial F_\theta}{\partial\theta} + F_r\right) + \frac{a_\phi}{r\sin\theta}\left(\frac{\partial F_\theta}{\partial\phi} - \cos\theta\,F_\phi\right)\right]\hat{\boldsymbol{\theta}}(\theta,\phi) + \left[a_r\frac{\partial F_\phi}{\partial r} + \frac{a_\theta}{r}\frac{\partial F_\phi}{\partial\theta} + \frac{a_\phi}{r}\left[\frac{1}{\sin\theta}\left(\frac{\partial F_\phi}{\partial\phi} + \cos\theta\,F_\theta\right) + F_r\right]\right]\hat{\boldsymbol{\phi}}(\phi)$$